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Solving Optimization Problem under Stochastic Max-Min Separable Linear Constraints

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Abstract— The max-min separable optimization problem is a special type of extremal algebra. It focuses on worst case for minimizing separable objective function with satisfied all max-min separable constraints to find the optimal solution. This problem was solved in previous studies by deterministic variables but in real situations some or all of the parameters could be described by stochastic variables. The aim of this paper is solving the problem under max-min stochastic linear constraints while keeping the decision variables as deterministic variables. The proposed algorithm solves that problem based on the concept of Monte Carlo method. The proposal algorithm can exclude almost unfeasible values of stochastic variables without need to enter the optimization process to avoid time consuming. The algorithm is effective in solving stochastic max-min optimization problems with max-min stochastic constraints, given any random variables distribution for all constraints or any individual distribution of each constraint. The results present the optimal solution in case of environmental stochastic using proposed algorithm. The best results from stochastic model are better than the optimal result from deterministic model. The present algorithm is efficient for different applications such as electricity network problems, transportation problems, supply chain and logistics problems which can be formulated as max-min separable optimization problem under max-min stochastic constraints.

Keywords— Max-Min separable function, Stochastic Max-Min separable constraints, Extremal Algebra, Monte Carlo method

I. INTRODUCTION

Max-min algebra is a one branch of Extremal Algebra which is a descriptive approach to describe the real life situations by combining max or min. Max-Min algebra and min-max algebra is maturity of max-plus and min-plus. In other words we can consider that max-min algebra or min-max algebra represent the worst case of min-function or max-function, respectively. This type of algebra was known as dioid algebra, semi-ring, idempotent and tropical algebra. Max-min Algebra was included in optimization problems as in the objective function and/or in the constraints.

In the following paragraphs present the history of max-min algebra and present the methods for handling problem and show the fields were max-min algebra is applied.

Cuninghame-Green presented the main concept of min-max algebra as a new type of algebra and presented all its algebraic properties[1],[2]. He also computed the exact value of the period for a given matrix[1]. Butkovic proposed max-Algebraand its links between system of linear equation, eignvalue-eignvector of problem and combinatorial optimization problem[3]. Tomášková presented Max-plus problem and its application in spreading of information [4]. Additionally, that type of algebra was presented as optimization problem with unimodel function in max-separable constraint. It was solved and applied in service-point-location problems as an application [5]. Zimmermann solved the same problem with disjunctive constraints [6]. Tharwat and Zimmermann suggested approach for solving optimization problems with a max-separable objective function and min-separable inequality constraints and applied that in location problems as a case study. Gavalec and his colleagues presented algorithms for solving optimization problems under (max, min)-linear equations and/or inequality constraints as a real number [7] . This type of algebra is applied in optimization problem as Max min include objective function and is presented as optimization problems under max-min separable equation and inequality constraints as a fuzzy number [8]. Other area of research called stochastic max-plus linear systems was solved by an approximation method to reduce the computational complexity in case of increase of stochastic variable [9]. Optimization of stochastic max–min-plus-scaling systems was solved by an approximation approach to reduce such computational complexity by using an approximation approach that is based on the moments of a random variable and that can be computed analytically[10]. Furthermore, max-min plus stochastic constraint was presented as machine time scheduling problems with stochastic processing time and was solved by using Monte Carlo Simulation[11]. It was also presented as service location problem solved by deterministic equivalence of chance constraint in case of stochastic variables were normally distributed [12]. All these studied could be applied max-min algebra in many applications such as gas-electricity systems interdependence [13], transportation problem. The main benefit of using max-min algebra in objective function and constraints is formulating the supply chine problem [14] and recourse allocation problem and other problems depending on concept of network optimization without needing traditional integer programming, linear programming, nonlinear programming and ….etc. all traditional method are representing that problem with multi number of constraints. Moreover, it can achieve multi objectives with the same priority by applying the max-min separable objective function.

From all of the previous attempts which focus on the necessity of solving the problem of Optimization problems under (max,min) - linear constraints in stochastic environment according to the situation needed. That problem was solved in fuzzy environment [8] but still not tested in case of uncertainty to observe the effect of stochastic variable with any distribution in the result of optimal solution.

This paper is organized as follows: Section 1 presents a review regarding the max-min optimization problem. Section 2 presents the background of the problem presented by Gavalec et al. [7]. Section 3 presents the developing algorithm for max-min separable problem under stochastic constraint using Monte Carlo method. Section 4 includes applying the proposal algorithm on a numerical example with stochastic right hand side constraint and another numerical example with stochastic left hand side constraint with keeping of deterministic decision variable. Finally, Section 5 concludes the effective stochastic variables in optimal solution and suggests future works to develop solving stochastic model.

II. MAX-MIN SEPARABLE PROBLEM GIVEN DETERMINISTIC CONSTRAINT MMSPDC

Gavalec et. al. presented the general form of an optimization problem with max-min separable objective function subject to max–min linear inequality constraints. They also illustrated the method for solving that problem and determining the set of feasible solution of the Max-Min Separable given Deterministic Constraint (MMSPDC) [7]

MMSPDC (M1) Model:

$$Max_j(Min(f_j(x_j))) \quad (1)$$

Subject to

$$Max_j\left(a1_{ij} \wedge x_j\right) \geq b1_i,\ i \in I1 \quad (2)$$

$$Max_j\left(a2_{ij} \wedge x_j\right) \leq b2_i,\ i \in I2 \quad (3)$$

$$lx_j \leq x_j \leq ux_j \quad (4)$$

M1

Where $f_j: R^m \to R^m$ are continuous functions, $x_j \in M$ are decision variables,

M is a feasible set of all solution, $x_j = (x_1, x_2, \ldots, x_m) \in R^m$, $a1_{ij}$ & $a2_{ij} \in R$, $b1_i \& b2_i \in R$, $\forall i \in I,\ j \in J$, $a1_{ij} \wedge x_j$ = *min*$(a1_{ij}, x_j)$, $a2_{ij} \wedge x_j$ = *min*$(a2_{ij}, x_j)$. Where *I1= {1,2,…,s}, I2 = {s+1, …,n}, J= {1, 2,…,m}, I = {1,2,..,n}, I = I1∪I2,(I1 for ≥constraints & I2 for ≤ constraints).*

The following lemmas investigate some properties of the feasible set of solutions M for inequality constraints proved by [7]. These lemmas helped us to determine the region of solutions and also determine the maximum point of feasible set of solutions in constraints (2) and (3). The Lemma 1 determines the feasible set of solution for less than or equal constraints (3) and (4). Moreover, Lemma 2 determines the feasible set of solution for greater than or equal constraint (2) considering of satisfying constraint (4) in both Lemma 1 and Lemma 2.

Lemma 1

The following lemma helped us to determine the region of solution by decreasing the upper bound of x_j and determining the $VL_{ij}^{\leq}$ which represent the set of solutions for the system of equation (3) and (4) let's define $VL_{ij}^{\leq}$ as follows:

$$VL_{ij}^{\leq} = \{x_j; (a2_{ij} \wedge x_j) \leq b2_i \text{ and } lx_j \leq x_j \leq ux_j \} \quad (5)$$

Therefore $VL_{ij}^{\leq}$ is the feasible set of the system and its upper bound represents the maximum element for the feasible set ***M*** satisfying (3) and (4)

For arbitrary i and j, $V_{ij}^{\leq}$ can be reformulated as follows:

$$VL_{ij}^{\leq} = \begin{cases} [lx_j, ux_j] & if\, a2_{ij} \leq b2_i \\ [lx_j, ux_j \wedge b2_i] & if\, a2_{ij} > b2_i \\ \emptyset & if\, a2_{ij} > b2_i \,\&\, b2_i < lx_j \end{cases} \quad (6)$$

Since $x_j \in M \Rightarrow x_j \in \bigcap_{i \in I_2} VL_{ij}^{\leq}\ \forall j \in J$,
Therefore $VL_{ij}^{\leq} = \emptyset \Leftrightarrow M = \emptyset$,
So, ux_{ij} can be defined as an upper bound of $VL_{ij}^{\leq} \neq \emptyset$ as a following:

$$ux_{ij} = \begin{cases} ux_j & a2_{ij} \leq b2_i \\ ux_j \wedge b2_i & a2_{ij} > b2_i \text{ and } b2_i \geq lx_j \end{cases} \quad (7)$$

Therefore the upper bound $ux_j^{max} = min_{i \in I2}\, ux_{ij}$ the element ux_j^{max} is the maximal element of feasible set of solution *M* satisfying constraints (3) and (4) and replacing (3)(4) by new upper bound ux_j^{max} ■

Remark: For any $x_j \in [lx_j, ux_j]$ if $x_j \leq ux_j^{max}$ we will have to find the solution for the system of equation (2) otherwise there is no solution for problem *M1*.

The following Lemma 2 determine maximum point for set of feasible solution ***M*** which satisfied the equation (2)

Lemma 2

The following lemma determines the set of feasible solution for greater than or equal constraint (2) with regard of satisfying the system of equation (2) and (4) by defining $V_{ij}^{\geq}$ as follows:

$$VG_{ij}^{\geq} = \{x_j : (a1_{ij} \wedge x_j) \geq b1_i \text{ and } lx_j \leq x_j \leq ux_j\} \quad (8)$$

Therefore $VG_{ij}^{\geq}$ is the feasible set of the system and its upper bound represents the maximum element for the feasible set ***M*** satisfying (2) and (4)

For arbitrary i and j, $VG_{ij}^{\geq}$ can be reformulated as follows:

$$VG_{ij}^{\geq} = \begin{cases} [b1_i \vee lx_j\, , ux_j] & if\ a1_{ij} \geq b1_i \text{ and } b1_i \leq ux_j \\ \emptyset & if\ a1_{ij} < b1_i \text{ or } b1_i > ux_j \end{cases} \quad (9)$$

Where $x_j \in M = \bigcap_{i \in I1} \bigcup_{j \in J} VG_{ij}^{\geq}, \forall j \in J$
Therefore $VG_{ij}^{\geq} = \emptyset \Leftrightarrow M = \emptyset$ ■

III. DEVELOPED ALGORITHM FOR MAX-MIN SEPARABLE PROBLEM UNDER STOCHASTIC CONSTRAINT USING MONTE CARLO METHOD

The following section presents the stochastic environment for MMSPDC model in two different cases, in the first one the stochastic parameters are considered in the left hand side of the constraint ($\tilde{a}1_{ij}$ and $\tilde{a}2_{ij}$),while in the second case the stochastic parameters are considered in the right hand side of the constraints ($\tilde{b}1_i$ and $\tilde{b}2_i$).

The model (M1) in first case where the left hand side of the constraint is stochastic variables ($\tilde{a}1_{ij}$ and $\tilde{a}2_{ij}$) can be formulated as follows:
M2 Model:

$$F(X) = Max_j(Min(f_j(x_j))) \quad (10)$$
Subject to
$$Max_j(\tilde{a}1_{ij} \wedge x_j) \geq b1_i, \qquad i \in I1 \quad (11)$$
$$Max_j(\tilde{a}2_{ij} \wedge x_j) \leq b2_i, \qquad i \in I2 \quad (12)$$
$$lx_j \leq x_j \leq ux_j \quad (13)$$
M2

Where $f_j: R^m \to R^m$ are continuous functions, $x_j \in \boldsymbol{M}$ are decision variables, $\boldsymbol{M}$ is a feasible set of solution, $x_j = (x_1, x_2, \dots, x_m) \in R^m$, $\tilde{a}1_{ij}$ and $\tilde{a}2_{ij}$ are random variables i.i.d with define distribution and given parameters (i.e. Normal distribution with given parameters mean and standard deviation such as $\tilde{a}1 \sim N(\mu_{a1}, \sigma_{a1})$, $\tilde{a}2 \sim N(\mu_{a2}, \sigma_{a2})$ where (μ_{a1}, μ_{a2}) are mean and (σ_{a1}, σ_{a2}) are standard deviation. $b1_i$ and $b2_i \in R$ $\forall$ $i \in I$, $j \in J$. , $I1 = \{1,2,\dots,s\}$, $I2 = \{s+1, \dots,n\}$, $I = I1 \cup I2$, I_1 for ≥ constraints and *I2* for ≤ constraints, $J = \{1, 2,\dots,m\}$, $I = \{1,2,..,n\}$.

Similarly, M2 for the second case where the right hand side of the constraint is a stochastic variable $\tilde{b}1_i$ and $\tilde{b}2_i$ can be formulated as follows:

M3 Model:
$$F(X) = Max_j(Min(f_j(x_j))) \quad (14)$$
Subject to
$$Max_j(a1_{ij} \wedge x_j) \geq \tilde{b}1_i \quad , \qquad i \in I1 \quad (15)$$
$$Max_j(a2_{ij} \wedge x_j) \leq \tilde{b}2_i \quad , \qquad i \in I2 \quad (16)$$
$$lx_j \leq x_j \leq ux_j \quad (17)$$
M3

$f_j: R^m \to R^m$ are continuous functions, $x_j \in \boldsymbol{M}$ are decision variables, $\boldsymbol{M}$ is a feasible set of solution, $x_j = (x_1, x_2, \dots, x_m) \in R^m$, $a1_{ij}$ and $a2_{ij} \in R$, $\tilde{b}1_i$ and $\tilde{b}2_i$ are random variables i.i.d with define distribution and given parameters (i.e. Normal distribution with given parameters mean and standard deviation such as $\tilde{b}1 \sim N(\mu_{b1}, \sigma_{b1})$, $\tilde{b}2 \sim N(\mu_{b2}, \sigma_{b2})$ where (μ_{b1}, μ_{b2}) are mean and (σ_{b1}, σ_{b2}) are standard deviation $\forall$ $i \in I, j \in J$, $I1 = \{1,2,\dots,s\}$, $I2 = \{s+1, \dots,n\}$, $I = I1 \cup I2$, I_1 for ≥ constraints and *I2* for ≤ constraints, $J = \{1, 2,\dots,m\}$.

In both models M2 and M3 the decision variables x_j are deterministic while the main concern of the two models is monitoring the changes of the optimal values and objective values according to the changes in the assumed stochastic variables.

A. Monte Carlo Method for Solving (M2 and M3) Models

Monte Carlo method is a numerical method for solving problems through generating random samples to obtain numerical result [15]. The difficulty of the Monte Carlo method is the extensive computational effort needed, but with the available of computers and through using pseudo-random number generator the effort and time have become much less. This method is a commonly method for representing stochastic variables and using in several application such as machine time scheduling[11], Estimate of the occlusion of 3D objects [16], simulation digital photon [17],…etc.
The Monte Carlo technique has several methods such as sampling the uniform distribution, inverse transform method and acceptance-rejection method. In the next algorithm focuses on Inverse transform method for solving M2 and M3 models.

The Inverse transform sampling starts with generating independent realizations of a random variable U uniformly distributed on [0,1] and using its inverse cumulative distribution $F^{-1}(X)$. Recall that the cumulative distribution for a random variable X is $F(X) = P(X \leq x)$ for specific probability distribution.
The following algorithm uses Lemma 1 and Lemma 2 to exclude the infeasible attempts of inverse transformation for random variables. These steps will be presented with details in the next algorithm

Algorithm (1): Monte Carlo for Max-Min Separable Constraint Algorithm (MCMMSCA)

Step 1: Enter name random variables and enter dimension of random variable (*i.e.* $\tilde{b}1^{n\times1}, \tilde{b}2^{n\times1}, \tilde{a}1^{k\times m}$ *or* $\tilde{a}2^{n-k\times m}$), and Identify distribution of the random variable (i.e. Normal distribution, Weibull distribution …etc.) and its corresponding parameters such as mean and variance in case of Normal Distribution, lx , ux and $b1$ in case of $\tilde{a}1$ are random variable.

Step 2: Generate probabilities following uniform distribution U [0,1], according to matrix size.

Step 3: Generate numbers for the variable using the probabilities from Step 2 and the inverse of the cumulative distribution function of the variable as: $\mathrm{RaNum}_{\mathrm{var}} = \mathrm{F}^{-1}(\mathrm{U})$ where $\mathrm{var} = \tilde{b}1, \tilde{b}2, \tilde{a}1$ *or* $\tilde{a}2$.

Step 4: While $\mathrm{RaNum}_{\tilde{b}1} \geq ux$ or $\mathrm{RaNum}_{\tilde{b}1} < 0$, go to Step 2 – Step 4 End
While $\mathrm{RaNum}_{\tilde{b}2} < lx$ or $\mathrm{RaNum}_{\tilde{b}2} < 0$, go to Step 2 – Step 4 End
While $\mathrm{RaNum}_{\tilde{a}1} < b1$ or $\mathrm{RaNum}_{\tilde{a}1} < 0$, go to Step 2 – Step 4 End
While $\mathrm{RaNum}_{\tilde{a}2} < 0$, go to Step 2 – Step 4 End

Step 5: Repeat Step 2 – Step 4 until all the required values of the random variable are positive and fall on feasible region of solution, Return $\mathrm{RaNum}_{\mathrm{var}}$.

After applying the previous algorithm (MCMMSCA**),** the number of unfeasible attempts will decrease due to excluding almost unfeasible values of stochastic variables without need to enter the optimization process in the following algorithm.

B. Monte Carlo Max-Min Stochastic Inequality Constraint MCMMSIC Algorithm:

The following algorithm applies Monte Carlo simulation in (MCMMSCA) algorithm to represent the stochastic variables in the optimization problem with inequality constraint of M2 and M3. In both cases, the number of constraints is represented *I1* and *I2* , Therefore, $I = I1 \cup I2$. Moreover, the algorithm allows choosing between the two cases of stochastic variable *(*$\tilde{a}1_{ij}$ and $\tilde{a}2_{ij}$*) or (* $\tilde{b}1_i$ and $\tilde{b}2_i$*).*

The algorithm for solving the max-min problem and stochastic variables in left hand side ($\tilde{a}1_{ij}$ and $\tilde{a}2_{ij}$) or right hand ($\tilde{b}1_i$ and $\tilde{b}2_i$) of the constraints using the (MCMMSCA) Algorithm as follows:

Algorithm (2): Monte Carlo Max-Min Stochastic Inequality Constraint Algorithm (MCMMSIC)

Step 1: Read $I_1, I_2,\ lx,\ ux$ and *F(X)*, deterministic variable (($a1$ & $a2$) or ($b1$ & $b2$) and parameter of random variables such as (mean, variance, in case of Normal distribution) where
$lx = (lx_1, lx_2, \dots, lx_m)$, $x = (ux_1, ux_2, \dots, ux_m)$ and $j = \{1, \dots, m\}$

Step 2: If ã2 random variable, apply **(MCMMSCA)** algorithm, see
Algorithm **(1**)
Set $\mathrm{a2} = \mathrm{RaNum}_{\tilde{a}2}$, End
If $\tilde{b}1$ and $\tilde{b}2$ are random variables, apply **(MCMMSCA)** algorithm, see
Algorithm **(1**).
Set $b1 = \mathrm{RaNum}_{\tilde{b}1}$, set $b2 = \mathrm{RaNum}_{b2}$, End

Step 3: for each $i \in I_2$ &$j \in J$
Find $VL_{ij}^{\leq}$ according to *Lemma 1*
At the end of this step the algorithm should determine ux which is maximum element of the set **M,** and satisfies all the $\leq$ constraints.

Step 4: if $\tilde{a}1$ is random variable apply **(MCMMSCA)** algorithm using
Algorithm (**1**),
set $a1 = \mathrm{RaNum}_{\tilde{a}1}$, End

Step 5: for each $i \in I_1$ &$j \in J$
Find $VG_{ij}^{\geq}$ according to *Lemma 2*
At the end of this step the algorithm should determine a new ux which is the maximum element of the set **M** and satisfies all constraints.

Step 6: Find $x_j^{(i)}$ for which $f_j\left(x_j^{(i)}\right) = min_{x_j^{(i)} \in VG_{ij}^{\geq}} f_j\left(x_j\right)$ where $VG_{ij}^{\geq} \neq \emptyset$ for all $i \in I_1$ &$j \in J$
Otherwise $f_j\left(x_j^{(i)}\right) = \infty$.

Step 7: Find $f_j(x_{j(i)}^{(i)}) = min\, f_j\left(x_j^{(i)}\right)$ for all $i \in I_1$ that represent the position of ***j*** in specific constraint ***i*** and has minimum of $f_j\left(x_j^{(i)}\right)$

Step 8: Find R_k the set of all $i \in I_1$ corresponding to *j(i)* that satisfy step 6
Therefore $R_k = \{i \in I_1; j(i) = k\} \forall k \in J$.

Step 9: Find T_k which represents the intersection of intervals $VG_{ik}^{\geq}$ for all $k \in j$ and $i \in R_k$.
Therefor $T_k = \bigcap_{i \in R_k} VG_{ik}^{\geq}\ \forall k \in J$ which is the feasible interval and includes an optimal solution for the problem

Step 10: Find $\hat{x}_k$according to
If $R_k = \emptyset$
then $f_k(\hat{x}_k) = min_{x_k \in T_k}(f_k(x_k))$ for all $k \in J$
Else $f_k(\hat{x}_k) = min_{x_k \in [lx_k, ux_k]}(f_k(x_k))$ for all $k \in J$

Step 11: Optimal solution is $\hat{x}$ and calculate objective function *F(x)*, Stop.

Algorithm (2): Pseudo code of (MCMMSIC) algorithm

```
Read Objective Function F(X), j ∈ J, i ∈ I, lx_j, and ux_j
If ã2 is random variable
Then
Enter name of distrbution and its parameter ,
set a2 = RaNum_ã2 ,
End if
If b̃1 and b̃2 are random variables
Then
Enter name of distrbution and its parameter ,
set b1 = RaNum_b̃1 and b2 = RaNum_b̃2
End if
For all i ∈ I_2, j∈ J Find VL^≤_ij
      If VL^≤_ij = ∅,
     Then M = ∅
     Print "No Feasible solution", Stop
    End if
End for
      For all i ∈ I_1, j∈ J Find VG^≥_ij
      If VG^≥_ij = ∅,
Then M = ∅
Print "No Feasible solution" ,Stop
End if
End for
ux is the maximum element of set M for all J
For i ∈ I_1 ; j ∈ J
      Find x_j^(i) where f_j(x_j^(i)) = min_{x_j∈VG^≥_ij}(f_j(x_j)) forall i ∈ I_1
      Find f_j(i)(x_j(i)^(i)) = min_{j∈J}(f_j(x_j^(i))) forall i ∈ I_1
      Find R_k = {i ∈ I_1; j(i) = k} ∀k ∈ J
      Find T_k = ∩_{i∈R_k} VG^≥_ik ∀k ∈ J
      If R_k = ∅
      Then x̂_k where f_k(x̂_k) = min_{x_k∈T_k}(f_k(x_k)) for all k ∈ J
      Else x̂_k where f_k(x̂_k) = min_{x_k∈[lx_k,ux_k]}(f_k(x_k))
      End if
End
      Set x* = x̂ that is optimal solution ,
      Calculate Objective Function F(X) of x* , Stop
```

IV. NUMERICAL EXAMPLES AND RESULTS

The max-min separable function under max-min constraint was applied in transportation problem as a follows: let's assume the trucks start journey from cities i where $i = \{1, \dots, n\}$ to reach local port *j* where *j* = *{1,2,...,m}* with road capacity a_{ij} where $a_{ij} = a1_{ij}$ and $a2_{ij}$, $a_{ij} \in R$. We have to connect that road to reach the terminal port T. Therefore, we need to select road capacity x_j as decision variables to travel truck from local ports j to terminal port (*T*), Then the road capacity from cities *i* to terminal port *T* via local port *j* is equal to $a1_{ij} \wedge x_j = min(a1_{ij}, x_j)$. The total capacity from *i* to *T* via *j* will be selected under at least one greater than or equal to given threshold $b1_i \in R$. On the other road capacity from city *i* to terminal port *T* via local port *j* is equal to $a2_{ij} \wedge x_j = min(a2_{ij}, x_j)$. The total capacity from *i* to *T* via *j* that will be selected under at least one less than or equal to given threshold $b2_i \in R$ and chosen capacity x_j between finite interval $x_j \in [lx_j, ux_j]$ where $lx_j, ux_j \in R$. Therefore the feasible vector of capacity $x = (x_1, x_2, \dots, x_m)$ must satisfy the constraints. Moreover, we need to choose x_j to optimize the object function in pessimistic case (maximum of minimum cost function). The capacities x_j are decision variables that affect the objective function such as cost function, delivery cost and taxes. This example is a fiction example to represent the meaning of numerical example [18]. See, Fig. 1.

In real situations $\tilde{a}_{ij}$ *or* $\tilde{b}_i = \tilde{b}1_i \& \tilde{b}2_i$ may be stochastic variables with definite continuous distribution and its parameters. The following, numerical examples were solved in two cases $\tilde{a}_{ij}$ or $\tilde{b}_i$ random variable with i.i.d using MCMMSC depending on Monte Carlo method for solving stochastic parameters with different constraint inequality (≥ and ≤).

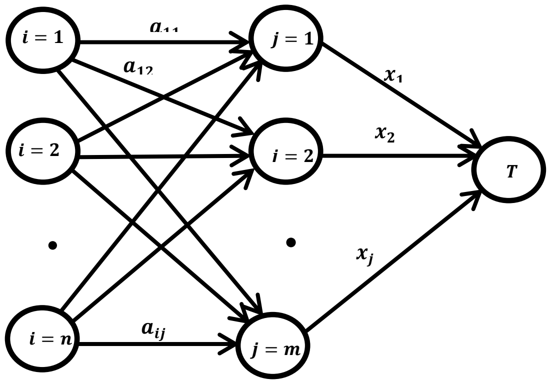

Fig. 1: The transportation problem was formulated by max-min separable objective function with max-min constraint.

The following numerical example P1 has stochastic variable in right hand side of max-min constraints (b1&b2) with normal distribution and given mean and standard deviation.

P1 :

$F(X) = F(x_1, x_2, x_3) = max(min(f_1(x_1)), min(f_2(x_2)), min(f_3(x_3)))$

Where $f_j(x_j) = |c_j x_j - d_j|$, $c = (3 \quad 3.3 \quad 6.5)$, $d = (7.2 \quad 15.9 \quad 9.2)$

s.t

$max(a1_{ij} \wedge x_j) \geq \tilde{b}1, \qquad i \in I_1$

$max(a2_{ij} \wedge x_j) \leq \tilde{b}2, \qquad i \in I_2$

$lx_j \leq x_j \leq ux_j$, $lx = (0 \quad 0 \quad 0)$, $ux = (10 \quad 10 \quad 10)$

$$a1 = \begin{pmatrix} 11 & 5 & 5 \\ 6.9 & 6.7 & 8.4 \\ 6 & 5.4 & 11.9 \\ 4.5 & 5.8 & 8.75 \\ 6.45 & 6.96 & 7.3 \end{pmatrix}, \tilde{b}1 \sim N(\mu_{b1}, \sigma_{b1}) \text{and}, \mu_{b1} = \begin{pmatrix} 2.698 \\ 4.22 \\ 3.895 \\ 1.98 \\ 4.135 \end{pmatrix}, \sigma_{b1} = \begin{pmatrix} 0.278 \\ 0.63 \\ 0.526 \\ 0.36 \\ 0.675 \end{pmatrix}$$

$$a2 = \begin{pmatrix} 5.5 & 15.5 & 8.3 \\ 3.3 & 4.8 & 7.7 \\ 14 & 7.3 & 10 \\ 8 & 8 & 13 \end{pmatrix}, \tilde{b}2 \sim N(\mu_{b2}, \sigma_{b2}) \text{ and } \mu_{b2} = \begin{pmatrix} 8.87 \\ 8.81 \\ 9.84 \\ 10.499 \end{pmatrix}, \sigma_{b2} = \begin{pmatrix} 0.955 \\ 0.71 \\ 1.159 \\ 1.32 \end{pmatrix}$$

Where $j = (1,2,\ldots,3)$, $I_1 = (1, \ldots, 5)$, $I_2 = (6, \ldots, 9)$ and $I = I_1 \cup I_2$

The following numerical example P2 is similar to P1 in objective function with stochastic variable in left hand side of max-min constraints $\tilde{a}1$and $\tilde{a}2$ with normal distribution given mean μ_{a1}, μ_{a2} and σ_{a1}, σ_{a2} in the previous example take $\mu_{a1}$and μ_{a2} equal to $a1$ and $a2$ from example P1, respectively. Moreover, take $b1$and $b2$ equal to $\mu_{b1}$and μ_{b2} from example P2, respectively.

P2 :

$f(x_1, x_2, x_3) = max(min(f_1(x_1)), min(f_2(x_2)), min(f_3(x_3)))$

Where $f_j(x_j) = |c_j x_j - d_j|$, $c = (3 \quad 3.3 \quad 6.5)$, $d = (7.2 \quad 15.9 \quad 9.2)$

s.t

$max(\tilde{a}1_{ij} \wedge x_j) \geq b1, \qquad i \in I_1$

$max(\tilde{a}2_{ij} \wedge x_j) \leq b2_i, \qquad i \in I_2$

$lx_j \leq x_j \leq ux_j$, $lx = (0 \quad 0 \quad 0)$, $ux = (10 \quad 10 \quad 10)$

$$\tilde{a}1 \sim N(\mu_{a1}, \sigma_{a1}), \quad \mu_{a1} = a1, \sigma_{a1} = \begin{pmatrix} 2.477 & 0.7228 & 1.083 \\ 1.23 & 1.265 & 1.827 \\ 1.413 & 0.88 & 1.657 \\ 1.77 & 2.33 & 1.88 \\ 0.749 & 1.256 & 1.77 \end{pmatrix} \text{ and } b1 = \mu_{b1}$$

$$\tilde{a}2 \sim N(\mu_{a2}, \sigma_{a2}), \quad \mu_{a2} = a2, \sigma_{a2} = \begin{pmatrix} 2.5 & 1.732 & 1.612 \\ 1.39 & 1.59 & 1.7 \\ 2.128 & 1.944 & 1.837 \\ 1.946 & 1.64 & 1.67 \end{pmatrix} \text{ and } b2 = \mu_{b2}$$

Where $j = (1,2,\ldots,3)$, $I_1 = (1,2,\ldots,5)$, $I_2 = (6,\ldots,9)$ and $I = I_1 \cup I_2$

The proposed Algorithms applied using software is Matlab R2013a and hardware machine is Intel (R) core™i5 4590M CPU. The algorithm MCMMSC is applied on case of normal distribution of all independent random variables.

The objective functions $F(X) = max(min(f_j(x_j))$ represent the worst case of $min(f_j(x_j))$ so the best solution that has minimum of $\max(\min(f_j(x_j))$ which represents the best result of algorithm. The presented algorithm applied in numerical example P1 and P2 and the result after 100, 500, 1000 and 10000 experiments. The result after 10000 experiments in case of $\tilde{a}1$ and $\tilde{a}2$ random variables more accurate than the other result found in the solution between 18.23 and 7.538E-06 ≅0 as maximum and best result respectively. The result of deterministic model is 1.023e-04 and the best result from stochastic model is 7.538e-06. The probability of improving the optimal solution from stochastic model is between 92.4% and 94% in P1. See, Table 1and Fig. **2**

On the other hand, the result after 10000 experiments in case of $\tilde{b}1$ and $\tilde{b}2$ random variables more accurate than the other result which found in the solution between 13.09 and 3.942e-06 ≅0 as maximum and best result respectively. The result of deterministic model is 1.023e-04 and the best result from stochastic model is 3.942e-06. The probability of improving the optimal solution from stochastic model is between 57% and 62.4% in P1. See Table 1 and Fig. **2** See Table 1 and Fig. **3**.

After solving the P2 by using proposal algorithm, the results show the maximum level of objective function decreases the region the objective function, see Fig. **2**and Fig. **3**. Moreover, the averages of objective functions are 0.3951 and 0.3742 in case of P1 and P2, respectively. So, we conclude the constraints with stochastic parameters in right hand side are more effective than the stochastic constraints in left hand side on the optimal solutions and objective functions, see Table 2 .

Table 1: Shows The Results after 100-500-1000 and 10000 Feasible Attempts

Example	Deterministic result	feasible attempts	Maximum	Best Result	Average F(X*)	Standard Deviation	Improvement %
P1	1.023e-04	100	5.46	7.538e-06	0.2861	1.142	94%
		500	16.12	7.538e-06	0.4776	1.557	90.2%
		1000	16.12	7.538e-06	0.3885	1.388	92.1%
		10000	18.23	7.538e-06	0.3951	1.504	92.4%
P2		100	4.071	4.116e-06	0.4009	0.8077	57%
		500	9.782	6.759e-06	0.3882	0.8874	62.4%
		1000	9.838	3.942e-06	0.3742	0.8802	58.9%
		10000	13.09	3.942e-06	0.3899	0.8872	60%

Table 2: Shows The Decision Variables $\mathbf{x}^*$ in Maximum Result and Best Result after 100-500-1000 and 10000 Feasible Attempts

Example	feasible attempts	Optimal solution in Maximum result $x^* = [x_1, x_2, x_3]$	Optimal solution in Best result $x^* = [x_1, x_2, x_3]$
P1	100	x^*= [4.22,4.82,1.42]	x^*= [2.4,4.82,1.42]
	500	x^*= [2.4,4.82, 3.895]	x^*= [2.4,4.82,1.42]
	1000	x^*= [2.4,4.82,3.895]	x^*= [2.4,4.82,1.42]
	10000	x^*= [2.4, 4.82,4.22]	x^*= [2.4,4.82,1.42]
P2	100	x^*= [2.4,5.74,1.42]	x^*= [2.4,4.82,1.42]
	500	x^*= [5.468,4.8,1.415]	x^*= [2.4,4.82,1.42]
	1000	x^*= [5.68,4.82,1.415]	x^*= [2.4,4.82,1.42]
	10000	x^*= [6.76,4.82,1.415]	x^*= [2.4,4.82,1.42]

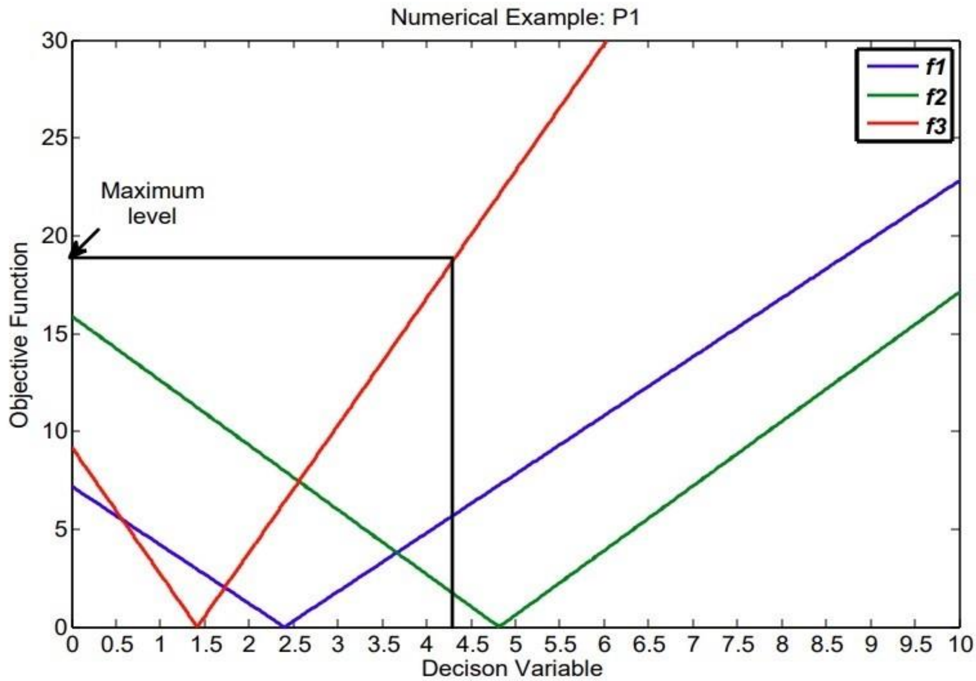

Fig. 2: Shows maximum level of objective functions are *f1* = $\mathbf{f_1}$ $(\mathbf{x_1})$, *f2* = $\mathbf{f_2}$ $(\mathbf{x_2})$ and *f3* = $\mathbf{f_3}$ $(\mathbf{x_3})$ after 10000 feasible attempts in P1 in case of stochastic $\tilde{\mathbf{a}}\mathbf{1}$ and $\tilde{\mathbf{a}}\mathbf{2}$

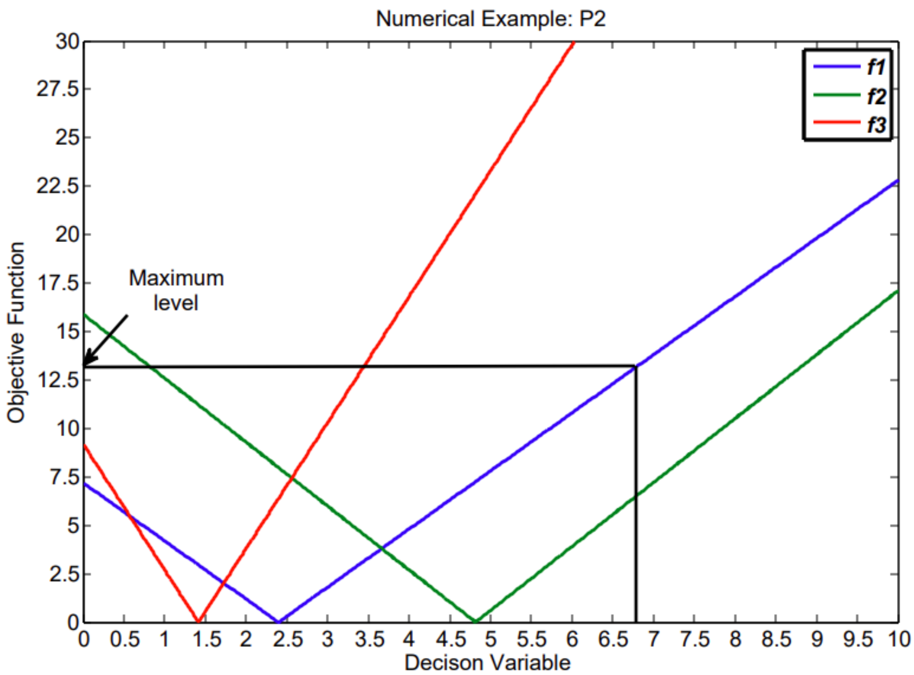

Fig. 3 : Shows maximum level of objective functions are *f1*=$\mathbf{f_1}$ $(\mathbf{x_1})$, *f2*= $\mathbf{f_2}$ $(\mathbf{x_2})$ and *f3*= $\mathbf{f_3}$ $(\mathbf{x_3})$ after 10000 feasible attempts in P2 in case of stochastic $\tilde{b}\mathbf{1}$ and $\tilde{b}\mathbf{2}$

V. CONCLUSION & FUTURE WORK

The Monte Carlo method was applied to solve the Max-Min Separable Function under Stochastic Constraints optimization problem (MMSFSC). The MCMMSC algorithm solves (MMSFSC) optimization problem which has inequality constraints only. The MCMMSC algorithm was developed based on the algorithm proposed in [19] and Monte Carlo method.

The proposal algorithms (MCMMSCA) and (MCMMSC) which are integrated to solve the max-min separable objective function was optimized under stochastic max-min constraints and tested through two numerical examples (P1 and P2). The proposal algorithm started by applying the (MCMMSCA) to exclude the infeasible attempts before entering the optimization process included in (MCMMSC). The (MCMMSC) algorithm was applied on stochastic variable in left hand side of constraint ($\tilde{a}1$ and $\tilde{a}2$) . Also, it was applied on stochastic variable in right hand side of constraint ($\tilde{b}1$ and $\tilde{b}2$) with defined distribution "normal distribution" and given its mean and standard deviation. The proposal algorithm was repeated in 100, 500, 1000 and 10000 feasible experiments to show the solution in different experiments. The several experiments were proved to be a good enough solution which works without accurate data and define range of solution. Moreover, the wide range of possible solution was covered by using high number of feasible 10000 attempts. The MCMMSC algorithm is effective and can be improved the optimal solution from the deterministic model especially in pessimistic case such as max-min stochastic separable function.

In future work, the proposed problem will be solved by using Monte-Carlo method with undefined distribution of random variables in constraints. Moreover, the proposal problem will be solved by using meta-heuristics methods such as particles swarm optimization, ant-colony optimization, bee colony optimization… etc. and comparing the result with proposed Algorithm to solve that problem with inequality constraints.

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